How does the *Fortune's Formula*-Kelly capital growth model perform?*

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Abstract

William Poundstone's (2005) book, *Fortune's Formula*, brought the Kelly capital growth criterion to the attention of investors. But how do full Kelly and fractional Kelly strategies that blend with cash actually preform in practice? To investigate this we revisit three simple investment situations and simulate the behavior of these strategies over medium term horizons using a large number of scenarios. These examples are from Bicksler and Thorp (1973) and Ziemba and Hausch (1986) and we consider many more scenarios and strategies. The results show:

1. the great superiority of full Kelly and close to full Kelly strategies over longer horizons with very large gains a large fraction of the time;
2. that the short term performance of Kelly and high fractional Kelly strategies is very risky;
3. that there is a consistent tradeoff of growth versus security as a function of the bet size determined by the various strategies; and
4. that no matter how favorable the investment opportunities are or how long the finite horizon is, a sequence of bad scenarios can lead to very poor final wealth outcomes, with a loss of most of the investor's initial capital.

Hence, in practice, financial engineering is important to deal with the short term volatility and long run situations with a sequence of bad scenarios. But properly used, the strategy has much to commend it, especially in trading with many repeated investments. [EFM Classification: 370]

*Dedicated to the memory of Kelly criterion pioneers John L. Kelly and Leo Breiman and Kelly critic Paul A. Samuelson.

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Introduction

In 1738 Daniel Bernoulli postulated that the marginal utility of an extra amount of money was proportional to the person's wealth. So $u'(w) = \frac{1}{w}$ where u is the investor's utility function, primes denote derivatives and w is wealth. Integrating gives $u(w) = \log w$. So log is the suggested utility function. Kelly (1956) and Latané (1959) argued that maximizing the expected utility with a log utility function is equivalent to the maximization of the long run growth rate. A derivation of this appears below. Breiman (1961) showed that the Kelly capital growth criterion had two long run properties. First, it maximizes the asymptotic long run growth rate. Secondly, it minimizes the time to achieve asymptotically large investment goals.

Discussion of various aspects of expected log maximization Kelly strategies and fractional Kelly strategies where one blends the Kelly strategy with cash are in the following papers: Browne (1998), Hakansson and Ziemba (1995), MacLean, Ziemba and Blazenko (1992), MacLean, Ziemba and Li (2005), McEnally (1986), Merton and Samuelson (1974), Mulvey, Pauling and Madey (2003), Rubinstein (1976, 1991), Samuelson (1969, 1971, 1979), Stutzer (2000, 2004), Thorp (2006, 2010), Wilcox (2003ab, 2005) and Ziemba (2010). Ziemba (2005) discusses the use of Kelly strategies by great investors such as Keynes, Buffett, Soros and others. Use of the Kelly strategies by *Morningstar* and *Motley Fool* are discussed by Fuller (2006) and Lee (2006). They follow Poundstone (2005) and fail to understand the multivariate aspects of multiple assets, short term risk, transaction costs and other features of these strategies. The book MacLean, Thorp and Ziemba (2010a) provides a fuller analysis of the advantages and disadvantages, theory and practice of these strategies and additional references. The Kelly capital growth model in the simplest case is derived as follows:

The asymptotic rate of asset growth is

$$G = \lim_{N\to\infty} log \left(\frac{w_N}{w_0}\right)^{\frac{1}{N}},$$

where w_0 is initial wealth and w_N is period N's wealth. Consider Bernoulli trials that win $+1$ with probability p and lose -1 with probability $1-p$. If we win M out of N of these independent trials, then the wealth after period N is

$$w_N = w_0(1+f)^M(1-f)^{N-M}$$

where f is the fraction of our wealth bet in each period. Then

$$G(f) = \lim_{N\to\infty} \left[\frac{M}{N} log(1+f) + \frac{N-M}{N} log(1-f)\right]$$

which by the strong law of large numbers is

$$G(f) = plog(1+f) + qlog(1-f) = E\log(w).$$

Hence, the criterion of maximizing the long run exponential rate of asset growth is equivalent to maximizing the one period expected logarithm of wealth. So, to maximize long run (asymptotic) wealth maximizing expected log is the way to do it period by period. See MacLean, Thorp and Ziemba (2010a) for papers that generalize these simple Bernoulli trial asset return assumptions.

The optimal fractional bet, obtained by setting the derivative of $G(f)$ to zero, is $f^* = p - q$ which is simply the investor's edge or the expected gain on the bet.[1] If the bets are win $O+1$ or lose 1, that is the odds are O to 1 to win, then the edge is $pO - q$ and the optimal Kelly bet is $f^* = \frac{pO-q}{O}$ or the $\frac{edge}{odds}$. Since the edge $pO - q$ is the measure of the mean and the odds is a risk concept, you wager more with higher mean and less with higher risk.

Observe that the bets can be very large. For example, if $p = .99$ and $q = .01$, the optimal bet is 98% of one's fortune! A real example of this by Mohnish Pabrai (2007), who won the bidding for the 2008 lunch with Warren Buffett paying more than $600,000, had the following investment in Stewart Enterprises as discussed by Thorp (2010). Over a 24-month period, with probability 0.80 the investment at least doubles, with 0.19 probability the investment breaks even and with 0.01 probability all the investment is lost. The optimal Kelly bet is 97.5% of wealth, half Kelly is 38.75%. Pabrai invested 10%. While this seems rather low, other investment opportunities, miscalculation of probabilities, risk tolerance, possible short run losses, bad scenario Black Swan events, price pressures, buying in and exiting suggest that a bet a lot lower than 97.5% is appropriate.

In general, Kelly bets are large and risky short term. One sees that from the Arrow-Pratt absolute risk aversion of the log utility criterion

$$R_A(w) = \frac{-u''(w)}{u'(w)} = 1/w$$

which is essentially zero for non-bankrupt investors. Hence, log can be an exceedingly risky utility function with wide swings in wealth values because the optimal bets can be so large.

[1] If there are two independent wagers and the size of the bets does not influence the odds, then an analytic expression can be derived, see Thorp (2006). In general, to solve for the optimal wagers in cases where the bets influence the odds, there is dependence, or for cases with three or more wagers, one must solve a non-convex nonlinear program, see Ziemba and Hausch (1987) for techniques. This gives the optimal wager taking into account the effect of our bets on the odds (prices).

Chopra and Ziemba (1993) investigated the effect of errors in mean, variance and covariance estimates in portfolio problems and show that the relative error impact is risk aversion dependent. For example, for typical problems there is a 20:2:1 ratio implying that errors in estimating the mean are ten times variance errors and twenty times co-variance errors in certainty equivalent value. But for log utility with extremely low risk aversion, this is more like 100:3:1. What's clear is that getting means correctly estimated is crucial for portfolio success.

Log utility is related to negative power utility, since with αw^{α} for $\alpha < 0$ since negative power converges to log when $\alpha \to 0$. So we can think of log as being the most risky and extreme negative power utility function. MacLean, Ziemba and Li (2005) have shown that the handy formula for the fraction $f = \frac{1}{1-\alpha}$ is optimal when the asset returns are lognormal. Here k is the Kelly strategy and $f \text{x} k$ is the fractional Kelly strategy. But, as Thorp (2010) has shown, this approximation can be poor if the assets are far from lognormal. Betting more than the Kelly amount leads to lower growth and more risk. That is linked to positive power utility which is to be avoided as it will invariably lead to disaster. See Ziemba and Ziemba (2007) for a discussion of some of these overbet disasters including LTCM, Niederhoffer and Amarath.

In continuous time, the long term optimal growth rate is

$$G^* = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 + r = \frac{1}{2}(\text{Sharpe Ratio})^2 + \text{ risk free asset},$$

where μ is the mean portfolio return, r is the risk free return and σ^2 is the portfolio return variance. So the ordinary Sharpe ratio determines the optimal growth rate.

A very important question is how much of an investor's wealth should be allocated to investments including cash and other assets using the Kelly or fractional Kelly criterions. The capital growth literature has optimal dynamic consumption-investment models such as Phelps (1962), Samuelson (1969), Hakansson (1970), etc. Ziemba-Vickson (1975, 2006) and MacLean, Thorp and Ziemba (2010a) discuss many of these models.

One way to look at this is to assume that there is consumption (c) and discretionary wealth ($w-c$). Wilcox (2003) has studied this and notes that discretionary wealth can be written in terms of an implicit leverage ratio $L = \frac{w}{w-c}$. The returns on risky assets are weighted or rescaled to reflect leveraging. The weighted return on a unit of capital invested is $1 + xLR$ where x and $1-x$ are the fractions in risky assets and risk free assets, respectively, and R is the random rate of return on the risky asset portfolio. The standard Kelly problem is $\max E_R \log(1 + xR)$ with Kelly optimal strategy weighting x^*. Wilcox's modification is $\max E \log(1 + xLR)$. So if x_w is a solution to this model - how does it relate to the Kelly weighting x^*? It is clear that $x_w = \frac{1}{L} x^*$ so the Wilcox discretionary portfolio is a fractional Kelly strategy.

This is not surprising since fractional Kelly strategies are a blend of the Kelly weighting and cash. The higher $L = \frac{w}{w-c}$ is the lower is the optimal Kelly fraction. Wilcox discusses some implications of the impact of leverage. Higher leverage investors are sensitive to fat tails in the returns distribution and will reduce the Kelly fraction strategy. In this paper we analyze the returns at the end of 40 periods and other horizons and show wealth trajectories and how the Kelly fraction affects those trajectories.

Motivation for this paper

The Kelly optimal capital growth investment strategy has many attractive long run theoretical properties. MacLean, Thorp and Ziemba (2010b) discuss the good and bad properties. It has been dubbed "fortunes formula" by Thorp (see Poundstone, 2005). However, the attractive long run properties are countered by negative short to medium term behavior based on conservative utility functions because of the almost zero Arrow-Pratt risk aversion of log utility. In this paper, the empirical performance of various Kelly and fractional Kelly strategies is considered in realistic investment situations. Three experiments from the literature (Ziemba and Hausch, 1986 and Bicksler and Thorp, 1973) are more fully examined with many more scenarios and strategies. The class of investment strategies generated by varying the fraction of investment capital allocated to the Kelly portfolio are applied to simulated returns from the models, and the distribution of final wealth is described.

Fractional Kelly Strategies: The Ziemba and Hausch Experiment

Consider an investment situation with five possible independent investments where one wagers \$1 and either loses it with probability $1 - p$ or wins $\$\,(O+1)$ with probability p, with odds 0 to 1. The five wagers with odds of $O = 1, 2, 3, 4$ *and* 5 to one all have the same expected value of 1.14. The optimal Kelly wagers in the one dimensional case are the expected value edge of 14% over the odds. Hence the wagers are from 14%, down to 2.8% of initial and current wealth at each decision point for the five investments. Table 1 describes these investments. The value 1.14 was chosen as it is the recommended cutoff for profitable place and show racing bets using the system described in Ziemba and Hausch (1986).

Ziemba and Hausch (1986) used 700 decision points and 1000 scenarios and compared full with half Kelly strategies. We use the same 700 decision points and 2000 scenarios and calculate more attributes of the various strategies. We use full, 3/4, 1/2, 1/4, and 1/8 Kelly strategies and compute the maximum, mean, minimum, standard deviation,

Win Probability	Odds	Prob of Selection in Simulation	Kelly Bets
0.570	1-1	0.1	0.140
0.380	2-1	0.3	0.070
0.285	3-1	0.3	0.047
0.228	4-1	0.2	0.035
0.190	5-1	0.1	0.028

Table 1: The Investment Opportunities

skewness, excess kurtosis and the number out of the 2000 scenarios for which the final wealth starting from an initial wealth of $1000 is more than $50, $100, $500 (lose less than half), $1000 (breakeven), $10,000 (more than 10-fold), $100,000 (more than 100-fold), and $1 million (more than a thousand-fold). Table 2 shows these results and illustrates the conclusions listed in the abstract. The final wealth levels are much higher on average, the higher the Kelly fraction. With 1/8 Kelly, the average final wealth is $2072, starting with $1000. It is $4339 with 1/4 Kelly, $19,005 with half Kelly, $70,991 with 3/4 Kelly and $524,195 with full Kelly. So as you approach full Kelly, the typical final wealth escalates dramatically. This is shown also in the maximum wealth levels which for full Kelly is $318,854,673 versus $6330 for 1/8 Kelly.

			Kelly Fraction		
Statistic	1.0k	0.75k	0.50k	0.25k	0.125k
Max	318854673	4370619	1117424	27067	6330
Mean	524195	70991	19005	4339	2072
Min	4	56	111	513	587
St. Dev.	8033178	242313	41289	2951	650
Skewness	35	11	13	2	1
Kurtosis	1299	155	278	9	2
$> 5 \times 10$	1981	2000	2000	2000	2000
10^2	1965	1996	2000	2000	2000
$> 5 \times 10^2$	1854	1936	1985	2000	2000
$> 10^3$	1752	1855	1930	1957	1978
$> 10^4$	1175	1185	912	104	0
$> 10^5$	479	284	50	0	0
$> 10^6$	111	17	1	0	0

Table 2: Final Wealth Statistics by Kelly Fraction for the Ziemba and Hausch Model

Figure 1 shows the wealth paths of these maximum final wealth levels. Most of the gain is in the final 100 of the 700 decision points. Even with these maximum graphs, there is much volatility in the final wealth with the amount of volatility generally higher with higher Kelly fractions. Indeed with 3/4 Kelly, there were losses from about decision points 610 to 670.

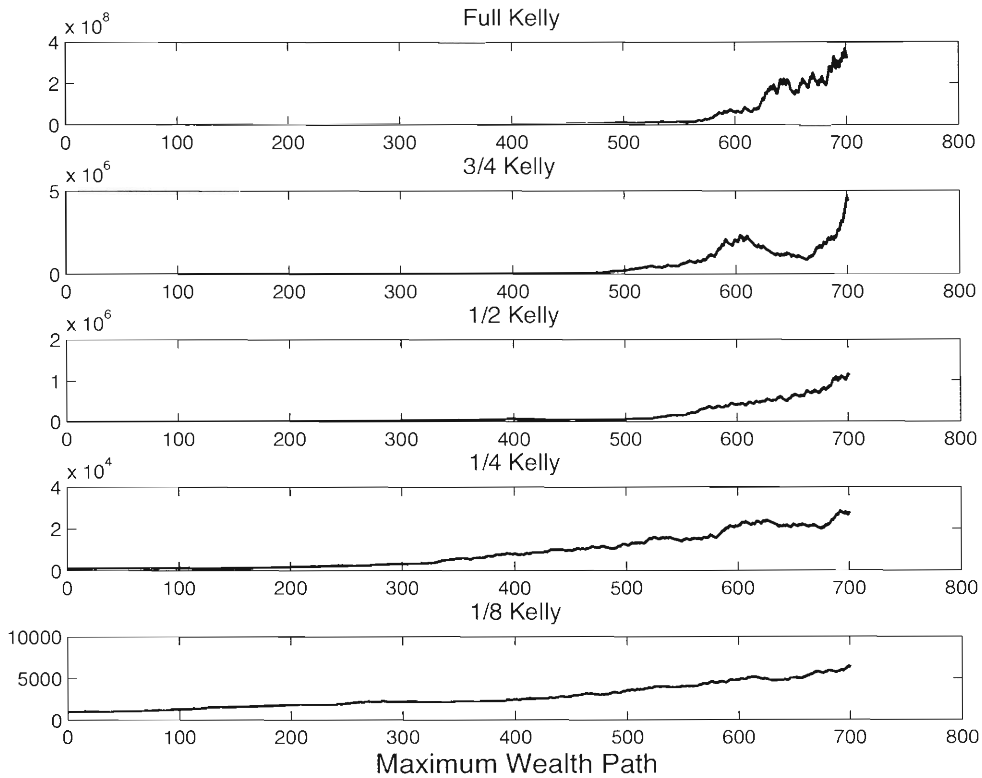


Figure 1: Highest Final Wealth Trajectory for the Ziemba and Hausch Model

Considering the chance of losses (final wealth is less than the initial \$1000) in all cases, even with 1/8 Kelly with 1.1% and 1/4 Kelly with 2.15%, there are losses with 700 independent bets each with an edge of 14%. For full Kelly, it is fully 12.4% losses, and it is 7.25% with 3/4 Kelly and 3.5% with half Kelly. These are just the percent of losses. But the size of the losses can be large as shown in the > 50, > 100, and > 500 rows of Table 2. The minimum final wealth levels were 587 for 1/8 and 513 for 1/4 Kelly so you never lose more than half your initial wealth with these lower risk betting strategies. But with 1/2, 3/4 and full Kelly, the minimums were 111, 56, and only \$4. Figure 2 shows these minimum wealth paths. With full Kelly, and by inference 1/8, 1/4, 1/2, and 3/4 Kelly, the investor can actually never go fully bankrupt because of the proportional nature of Kelly betting.

If capital is infinitely divisible and there is no leveraging, then the Kelly bettor cannot go bankrupt since one never bets everything (unless the probability of losing anything at all

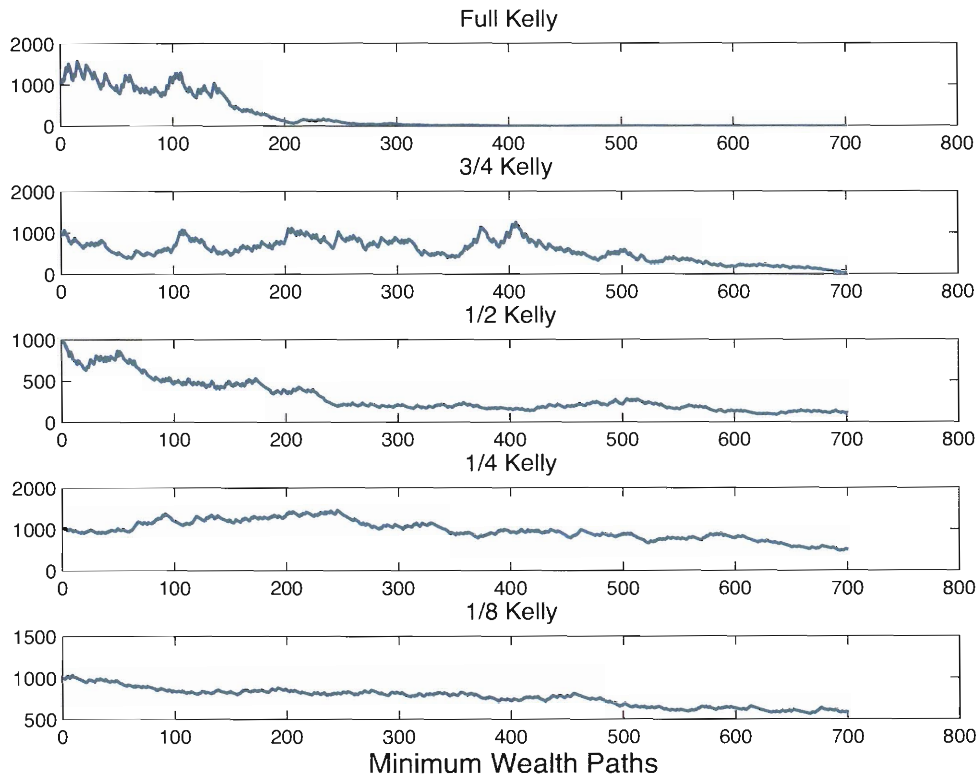


Figure 2: Lowest Final Wealth Trajectory for the Ziemba and Hausch Model

is zero and the probability of winning is positive). If capital is discrete, then presumably Kelly bets are rounded down to avoid overbetting, in which case, at least one unit is never bet. Hence, the worst case with Kelly is to be reduced to one unit, at which point betting stops. Since fractional Kelly bets less, the result follows for all such strategies. For levered wagers, that is, betting more than one's wealth with borrowed money, the investor can lose more than their initial wealth and become bankrupt.

Proportional Investment Strategies: Alternative Experiments

The growth and risk characteristics for proportional investment strategies such as the Kelly depend upon the returns on risky investments. We now consider two other investment experiments where the return distributions are quite different. The mean returns are similar: 14% for Ziemba-Hausch, 12.5% for Bicksler-Thorp Example I, and 10.2% for Bicksler-Thorp Example II. However, the variation around the mean is not similar and this produces much different Kelly strategies and corresponding wealth trajectories for scenarios. The third experiment involves two assets: risky stock and safe cash which can be levered to produce considerable losses as well as large gains.

The Ziemba and Hausch Model

The first experiment looks further at the Ziemba and Hausch (1986) model. A simulation was performed with 3000 scenarios over $T = 40$ decision points with the five types of independent investments for various investment strategies. The Kelly fractions and the proportion of wealth invested are in Table 3. Here, $1.0k$ is full Kelly, the strategy which maximizes the expected logarithm of wealth. Values below 1.0 are fractional Kelly and coincide approximately with the decision from using a negative power utility function. The approximation $f = \frac{1}{1-\alpha}$, where $f\mathrm{x}k$ is the fractional Kelly strategy and $\alpha < 0$ is from the negative power utility function αw^{α}, is exactly correct for lognormally distributed assets and approximately correct otherwise, see MacLean, Ziemba and Li (2005) for proof. But the approximation can be poor for asset distributions far from log normal; see Thorp (2010). Values above 1.0 coincide with those from some positive power utility function. This is overbetting according to MacLean, Ziemba and Blazenko (1992), because the long run growth rate falls and security (measured by the chance of reaching a specific positive goal before falling to a negative growth level) also falls. This is very important and not fully understood by many hedge fund and other investors. Long Term Capital's 1998 demise from overbetting is a prime example. Examples of other similar blowouts are discussed in Ziemba and Ziemba (2007). They argue that the recipe for hedge fund disaster is almost always the same: the portfolio is overbet and not well diversified in some scenarios and then a bad scenario occurs in a non-diversified part of the portfolio and disaster quickly follows. The trouble is that the penalty for losses, especially with large portfolios in the billions is way too low. Fees are kept and fired traders either retire or get rehired by other funds. Examples include John Merriwether, Victor Niederhoffer and Brian Hunter.

				Kelly Fraction: f			
Opportunity	*1.75k*	*1.5k*	*1.25k*	*1.0k*	*0.75k*	*0.50k*	*0.25k*
A	0.245	0.210	0.175	0.140	0.105	0.070	0.035
B	0.1225	0.105	0.0875	0.070	0.0525	0.035	0.0175
C	0.08225	0.0705	0.05875	0.047	0.03525	0.0235	0.01175
D	0.06125	0.0525	0.04375	0.035	0.02625	0.0175	0.00875
E	0.049	0.042	0.035	0.028	0.021	0.014	0.007

Table 3: The Investment Proportions (λ) and Kelly Fractions. The investment proportions λ's are full Kelly as in the 5^{th} column; see Table 1. The other proportions are scaled from the Kelly fractions f in this column

The initial wealth for investment was 1000. Table 4 reports final wealth statistics on the for $T = 40$ periods with the various strategies.

				Fraction			
Statistic	1.75k	1.5k	1.25k	1.0k	0.75k	0.50k	0.25k
Max	50364.73	25093.12	21730.90	8256.97	6632.08	3044.34	1854.53
Mean	1738.11	1625.63	1527.20	1386.80	1279.32	1172.74	1085.07
Min	42.77	80.79	83.55	193.07	281.25	456.29	664.31
St. Dev.	2360.73	1851.10	1296.72	849.73	587.16	359.94	160.76
Skewness	6.42	4.72	3.49	1.94	1.61	1.12	0.49
Kurtosis	85.30	38.22	27.94	6.66	5.17	2.17	0.47
$> 5 \times 10$	2998	3000	3000	3000	3000	3000	3000
10^2	2980	2995	2998	3000	3000	3000	3000
$> 5 \times 10^2$	2338	2454	2634	2815	2939	2994	3000
$> 10^3$	1556	11606	1762	1836	1899	1938	2055
$> 10^4$	43	24	4	0	0	0	0
$> 10^5$	0	0	0	0	0	0	0
$> 10^6$	0	0	0	0	0	0	0

Table 4: Wealth Statistics by Kelly Fraction for the Ziemba and Hausch Model

Since the Kelly bets are small, the proportion of current wealth invested is not high for any of the fractions. The upside and down side are not dramatic in this example, although there is a substantial gap between the maximum and minimum wealth with the highest fraction. Figure 3 shows the trajectories which have the highest and lowest final wealth for a selection of fractions. The log-wealth is displayed to show the rate of growth at each decision point. The lowest trajectories are almost a reflection of the highest ones.

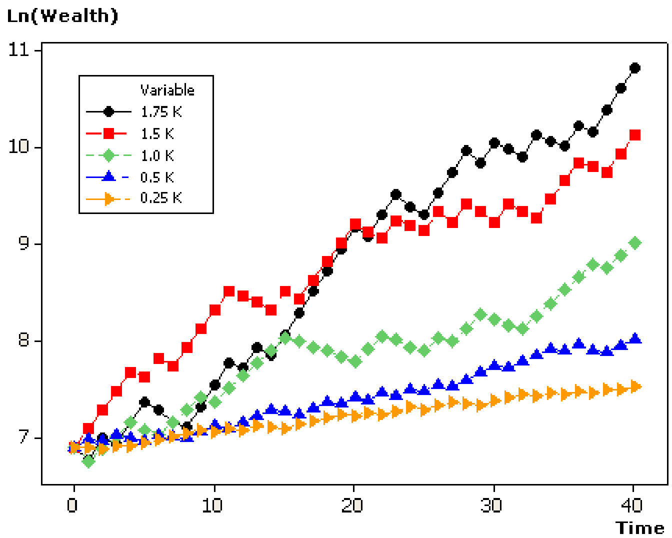
The skewness and kurtosis indicate that final wealth is not normally distributed. This is expected since the geometric growth process suggests a log-normal wealth. Figure 4 displays the simulated log-wealth for selected fractions at the horizon $T = 40$. The normal probability plot will be linear if terminal wealth is distributed log-normally. The slope of the plot captures the shape of the log-wealth distribution. For this example the final wealth distribution is close to log-normal. As the Kelly fraction increases the slope increases, showing the longer right tail but also the increase in downside risk in the wealth distribution.

On the inverse cumulative distribution plot, the initial wealth $\ln(1000) = 6.91$ is indicated to show the chance of losses. And indeed there can be considerable losses as shown in the left side of Figure 4a. The inverse cumulative distribution of log-wealth is the basis of comparisons of accumulated wealth at the horizon. In particular, if the plots intersect then first order stochastic dominance by a wealth distribution does not exist (Hanoch and Levy, 1969). The mean and standard deviation of log-wealth are considered in Figure 5, where the growth versus security trade-off by Kelly fraction is shown. The mean log-wealth peaks at the full Kelly strategy whereas the standard deviation is monotone increasing. Fractional strategies greater than full Kelly are inefficient in log-wealth, since the growth rate decreases and the standard deviation of log-wealth increases. It is these where the hedge fund disasters occur.

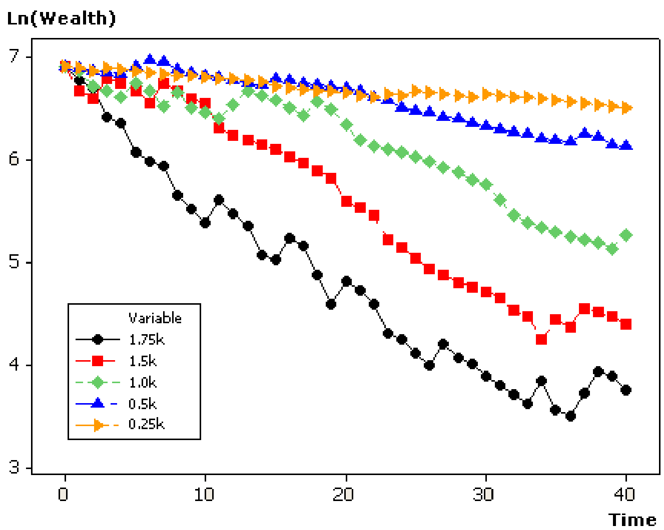
We have the following conclusions:

1. The statistics describing end of horizon ($T = 40$) wealth are all monotone in the fraction of wealth invested in the Kelly portfolio. The maximum terminal wealth and the mean terminal wealth increase in the Kelly fraction. In contrast the minimum wealth decreases as the fraction increases and the standard deviation grows as the fraction increases. There is a trade-off between wealth growth and risk. The cumulative distributions in Figure 4 supports the theory for fractional strategies, as there is no dominance, and the distribution plots all intersect.

2. The maximum and minimum final wealth trajectories show the wealth growth - risk trade-off of the strategies. The worst scenario is the same for all Kelly fractions so that the wealth decay is greater with higher fractions. The best scenario differs for the low fraction strategies, but the growth path is almost monotone in the fraction. The mean-standard deviation trade-off demonstrates the inefficiency of levered strategies (greater than full Kelly) which, as shown, are growth-security dominated and to be avoided in practice by responsible traders.

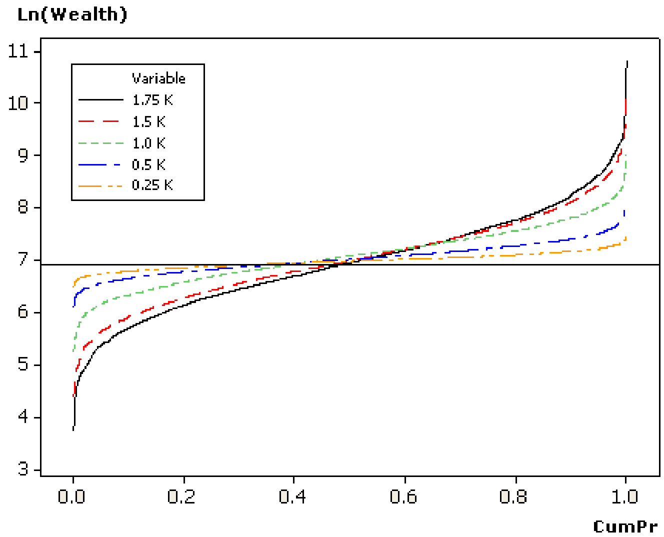


(a) Maximum

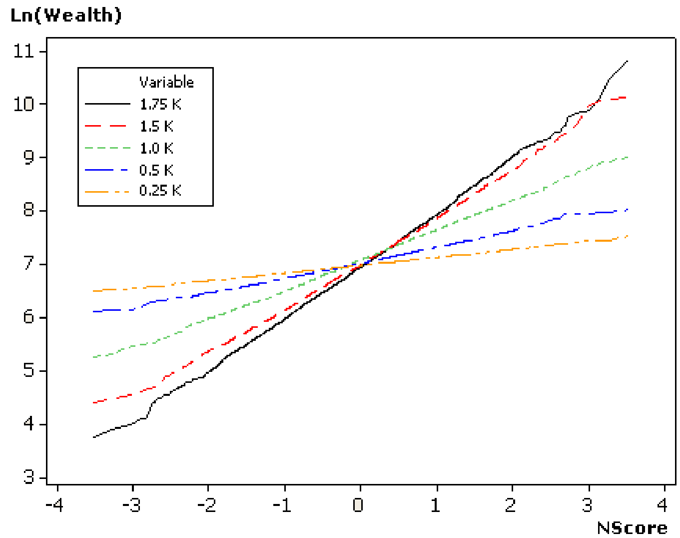


(b) Minimum

Figure 3: Trajectories with Final Wealth Extremes for the Ziemba and Hausch Model



(a) Inverse Cumulative



(b) Normal Plot

Figure 4: Final Ln(Wealth) Distributions by Fraction for the Ziemba and Hausch Model

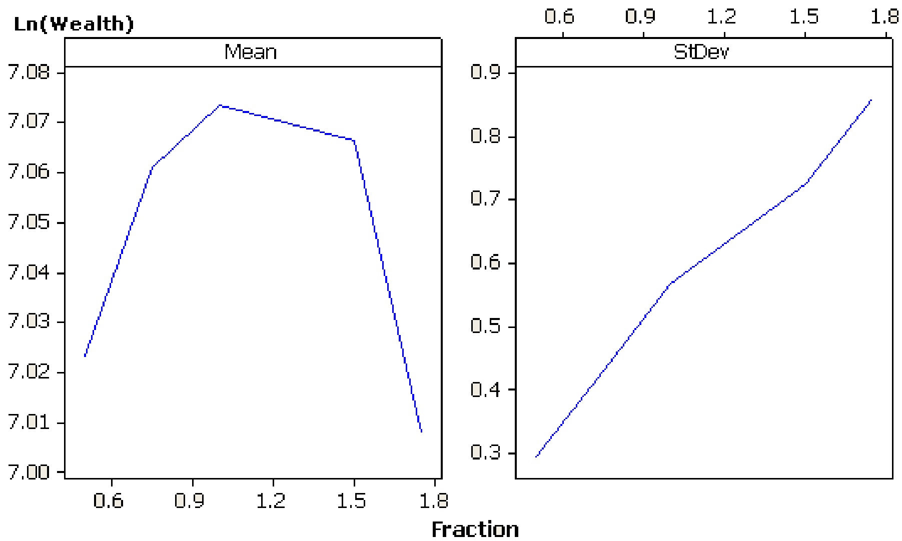


Figure 5: Mean-Std Tradeoff for the Ziemba and Hausch Model

Bicksler and Thorp Example I - Uniform Returns

There is one risky asset R with mean return of +12.5%, that is uniformly distributed between -25% and +50% for each dollar invested. Assume we can lend or borrow capital at a risk free rate $r = 0.0$. Let λ be the proportion of capital invested in the risky asset, where λ ranges from 0.4 to 2.4. So $\lambda = 2.4$ means \$1.4 is borrowed for each \$1 of current wealth and \$2.40 is invested in the risky asset. The Kelly optimal growth investment in the risky asset is $x = 2.8655$. The Kelly fractions for the different values of λ are shown in Table 5.[2] Bicksler and Thorp used 10 and 20 yearly decision periods, and 50 simulated scenarios. We use 40 yearly decision periods, with 3000 scenarios.

Proportion: λ	0.4	0.8	1.2	1.6	2.0	2.4
Fraction: f	0.140	0.279	0.419	0.558	0.698	0.838

Table 5: The Investment Proportions and Kelly Fractions for the Bicksler and Thorp Example I

The numerical results from the simulation with $T = 40$ are in Tables 6 - 8. Although the Kelly investment is levered, the fractions in this case are less than 1.

In this experiment the Kelly proportion is high, based on the attractiveness of the investment in stock. The largest fraction (0.838k) has high returns, although in the worst scenario most of the wealth is lost. The trajectories for the highest and lowest terminal wealth scenarios are displayed in Figure 6. The highest rate of growth is for the highest fraction, and correspondingly it has the largest wealth fallback.

The distribution of terminal wealth in Figure 7 illustrates the growth of the $f = 0.838k$ strategy. It intersects the normal probability plot for other strategies very early and increases its advantage. The linearity of the plots for all strategies is evidence of the log-normality of final wealth. The inverse cumulative distribution plot indicates that the chance of losses is small - the horizontal line indicates the log of initial wealth.

As further evidence of the superiority of the $f = 0.838k$ strategy consider the mean and standard deviation of log-wealth in Figure 8. The growth rate (mean ln(Wealth)) continues to increase since the fractional strategies are less then full Kelly. So there is no more than full Kelly over betting in this strategy.

[2]The formula relating λ and f for this experiment is as follows. For the problem

$$Max_x \left\{E(ln(1 + r + x(R - r)\right\},$$

where R is uniform on $[a, b]$ and r =the risk free rate. The first order condition

$$\int_a^b \frac{R - r}{1 + r + x(R - r)} \times \frac{1}{b - a} dR = 0,$$

reduces to

$$x(b - a) = (1 + r)ln\left(\frac{1 + r + x(b - r)}{1 + r + x(a - r)}\right) \Longleftrightarrow \left[\frac{1 + r + x(b - r)}{1 + r + x(a - r)}\right]^{\frac{1}{x}} = e^{\frac{b-a}{1+r}}.$$

For $a = -0.25, b = 0.5, r = 0$. The equation becomes

$$\left[\frac{1 + 0.5x}{1 - 0.25x}\right]^{\frac{1}{x}} = e^{0.75},$$

with a solution $x = 2.8655$.

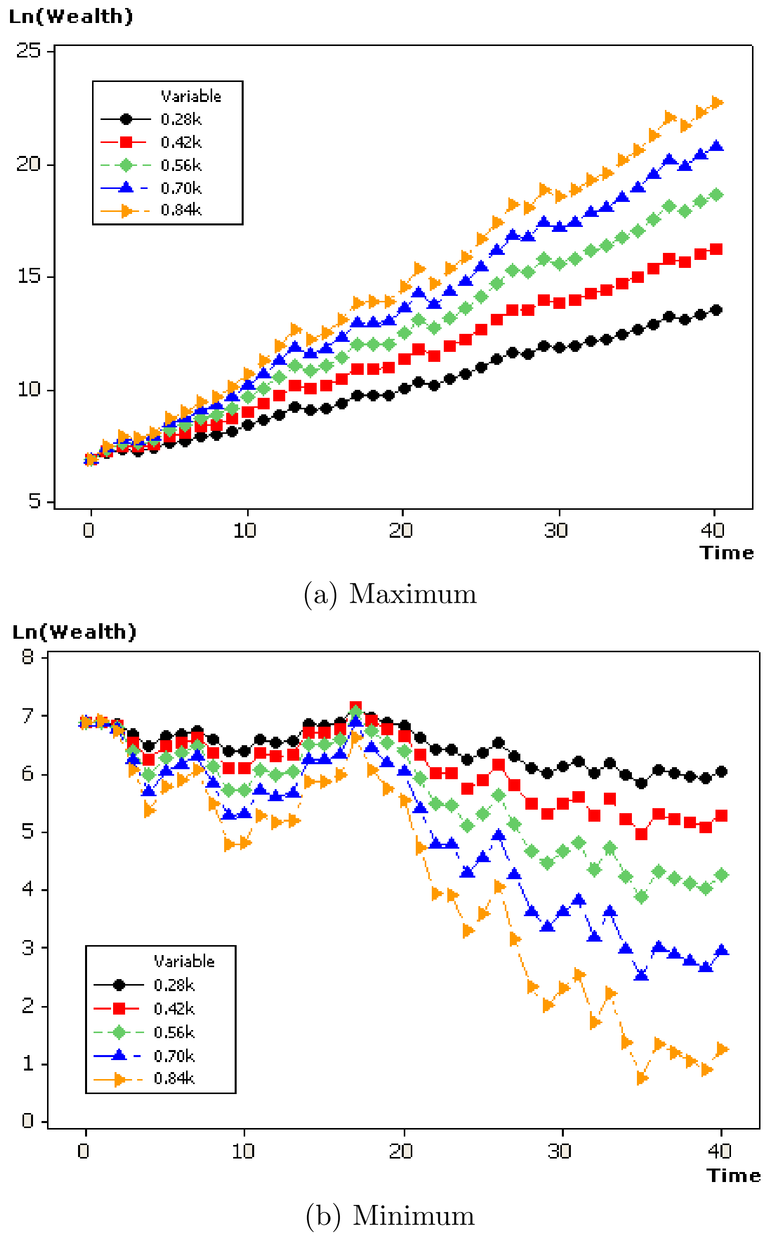
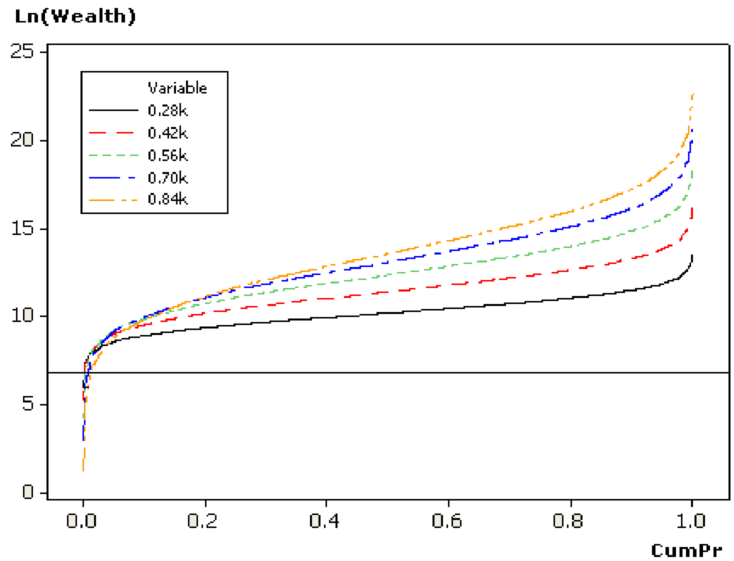
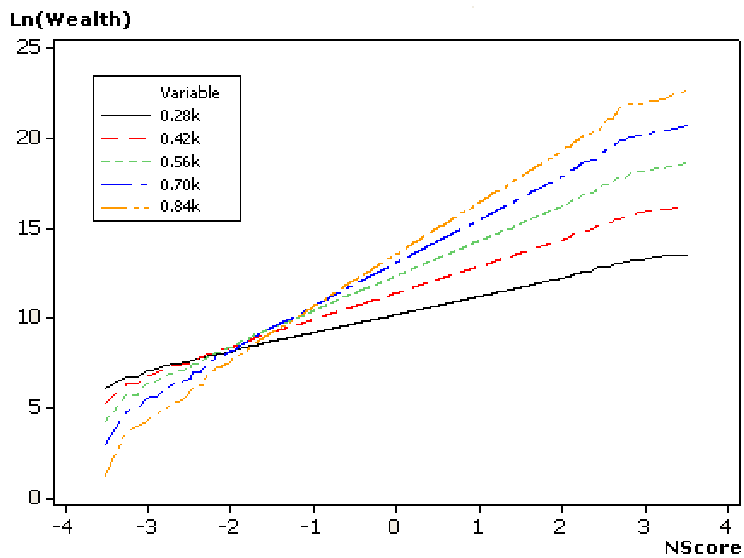


Figure 6: Trajectories with Final Wealth Extremes for the Bicksler and Thorp Example I



(a) Inverse Cumulative



(b) Normal Plot

Figure 7: Final Ln(Wealth) Distributions by Fraction: Bicksler-Thorp Example I

			Fraction			
Statistic	0.14k	0.28k	0.42k	0.56k	0.70k	0.84k
Max	34435.74	743361.14	11155417.33	124068469.50	1070576212.0	7399787898
Mean	7045.27	45675.75	275262.93	1538429.88	7877534.72	36387516.18
Min	728.45	425.57	197.43	70.97	18.91	3.46
St. Dev.	4016.18	60890.61	674415.54	6047844.60	44547205.57	272356844.8
Skewness	1.90	4.57	7.78	10.80	13.39	15.63
Kurtosis	6.00	31.54	83.19	150.51	223.70	301.38
$> 5 \times 10$	3000	3000	3000	3000	2999	2998
10^2	3000	3000	3000	2999	2999	2998
$> 5 \times 10^2$	3000	2999	2999	2997	2991	2976
$> 10^3$	2998	2997	2995	2991	2980	2965
$> 10^4$	529	2524	2808	2851	2847	2803
$> 10^5$	0	293	1414	2025	2243	2290
$> 10^6$	0	0	161	696	1165	1407

Table 6: Final Wealth Statistics by Kelly Fraction for the Bicksler and Thorp Example I

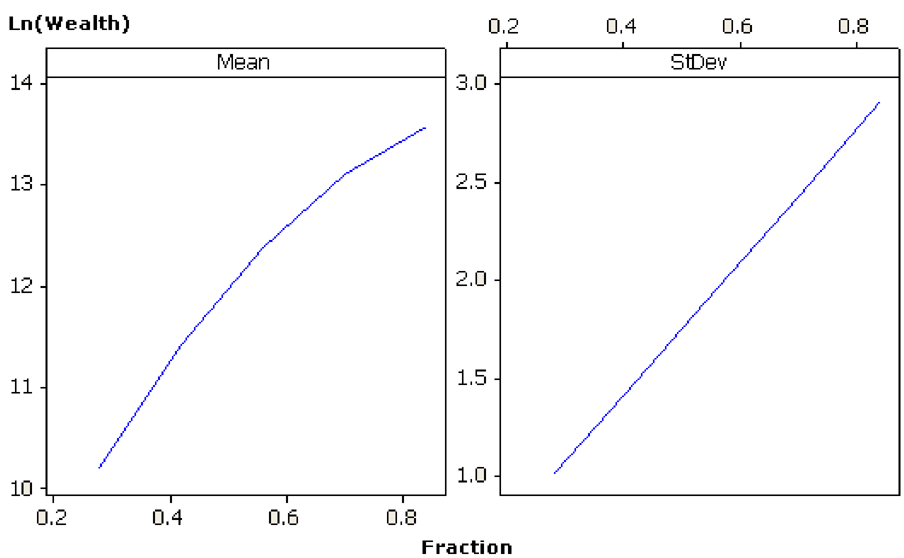


Figure 8: Mean-Std Trade-off for the Bicksler and Thorp Example I

We have the following conclusions:

1. The statistics describing end of horizon ($T = 40$) wealth are again monotone in the fraction of wealth invested in the Kelly portfolio. Specifically the maximum terminal wealth and the mean terminal wealth increase in the Kelly fraction. In contrast the minimum wealth decreases as the fraction increases and the standard deviation grows as the fraction increases. The growth and decay are much more pronounced then was the case in experiment 1. The minimum still remains above 0 since the fraction of Kelly is less than 1. There is a trade-off between wealth growth and risk, but the advantage of leveraged investment is clear. As illustrated with the cumulative distributions in Figure 7, the log-normality holds and the upside growth is more pronounced than the downside loss. Of course, the fractions are less than 1 so improved growth is expected.

2. The maximum and minimum final wealth trajectories clearly show the wealth growth - risk of various strategies. The mean-standard deviation trade-off favors the largest fraction, even though it is highly levered.

Bicksler - Thorp Example II - Equity versus Cash

In the third experiment there are two assets: US equities and US T-bills. According to Siegel (2002), during 1926-2001 US equities returned 10.2% with a yearly standard deviation of 20.3%, and the mean return was 3.9% for short term government T-bills with zero standard deviation. We assume the choice is between these two assets in each period. The Kelly strategy is to invest a proportion of wealth $x = 1.5288$ in equities and sell short the T-bill at $1 - x = -0.5228$ of current wealth. With the short selling and levered strategies, there is a chance of substantial losses. For the simulations, the proportion: λ of wealth invested in equities[3] and the corresponding Kelly fraction f are provided in Table 7. Bicksler and Thorp used 10 and 20 yearly decision periods, and 50 simulated scenarios. We use 40 yearly decision periods, with 3000 scenarios.

The results from the simulations with experiment 3 appear in Tables 8 - 11. The striking aspects of the statistics in Table 8 are the sizable gains and losses. For the most aggressive

[3]The formula relating λ and f for this example is as follows. For the problem

$$Max_x \left\{ E(ln(1 + r + x(R - r)) \right\},$$

where R is assumed to be Gaussian with mean μ_R and standard deviation σ_R, and r =the risk free rate. The solution is given by Merton (1990) as

$$x = \frac{\mu_R - r}{\sigma_R}.$$

Since $\mu_R = 0.102, \sigma_R = 0.203, r = 0.039$, the Kelly strategy is $x = 1.5288$.

λ	0.4	0.8	1.2	1.6	2.0	2.4
f	0.26	0.52	0.78	1.05	1.31	1.57

Table 7: Kelly Fractions for the Bicksler and Thorp (1973) Example II

strategy (1.57k), it is possible to lose 10,000 times the initial wealth. This assumes that the shortselling is permissable through to the decision period at the horizon T=40.

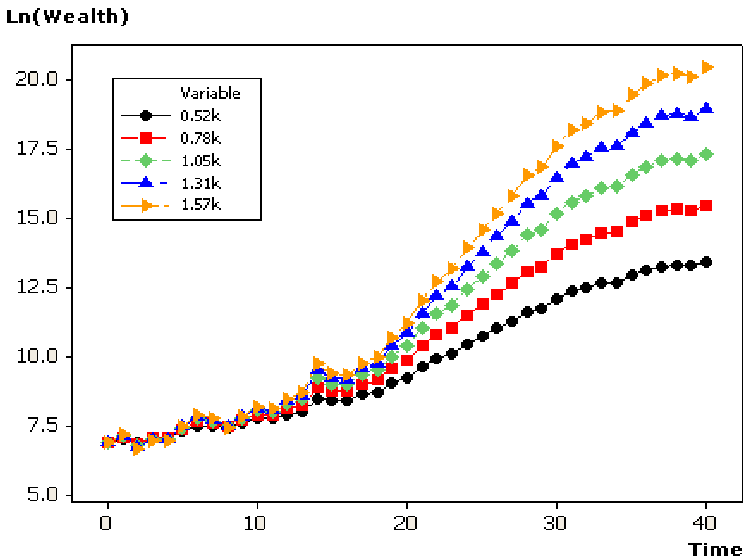
Table 8: Final Wealth Statistics by Kelly Fraction for the Bicksler and Thorp Example II

			Fraction			
Statistic	0.26k	0.52k	0.78k	1.05k	1.31k	1.57k
Max	65842.09	673058.45	5283234.28	33314627.67	174061071.4	769753090
Mean	12110.34	30937.03	76573.69	182645.07	416382.80	895952.14
Min	2367.92	701.28	-4969.78	-133456.35	-6862762.81	-102513723.8
St. Dev.	6147.30	35980.17	174683.09	815091.13	3634459.82	15004915.61
Skewness	1.54	4.88	13.01	25.92	38.22	45.45
Kurtosis	4.90	51.85	305.66	950.96	1755.18	2303.38
$> 5 \times 10$	3000	3000	2998	2970	2713	2184
10^2	3000	3000	2998	2955	2671	2129
$> 5 \times 10^2$	3000	3000	2986	2866	2520	1960
$> 10^3$	3000	2996	2954	2779	2409	1875
$> 10^4$	1698	2276	2273	2112	1794	1375
$> 10^5$	0	132	575	838	877	751
$> 10^6$	0	0	9	116	216	270

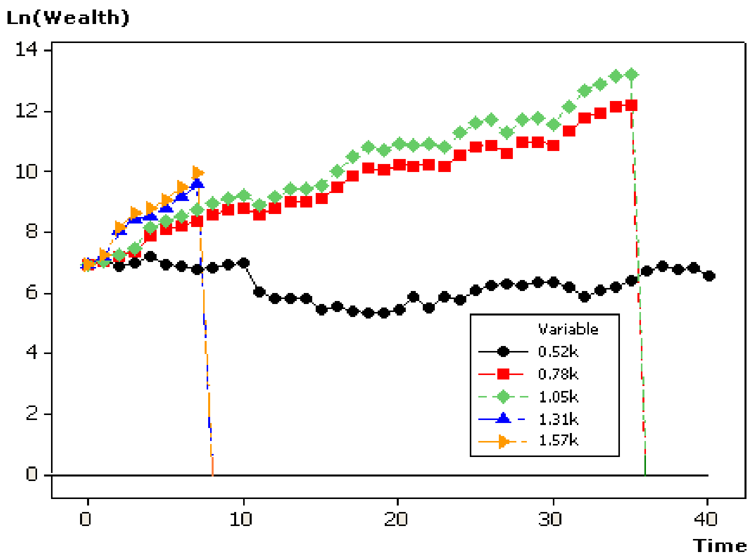
The highest and lowest final wealth trajectories are presented in Figure 9. In the worst case, the trajectory is terminated to indicate the timing of vanishing wealth. There is quick bankruptcy for the aggressive strategies.

The substantial downside is further illustrated in the distribution of final wealth plot in Figure 10. The normal probability plots are almost linear on the upside (log-normality), but the downside is much more extreme than log-normal for all strategies except for 0.52k. Even the full Kelly is very risky in this example largely because the basic position is levered. The inverse cumulative distribution shows a high probability of large losses with the most aggressive strategies. In constructing these plots the negative growth was incorporated with the formula $growth = [signW_T]\, ln(|W_T|)$.

The mean-standard deviation trade-off in Figure 11 provides more evidence concerning the riskyness of the high proportion strategies. When the fraction exceeds the full Kelly, the drop-off in growth rate is sharp, and that is matched by a sharp increase in standard

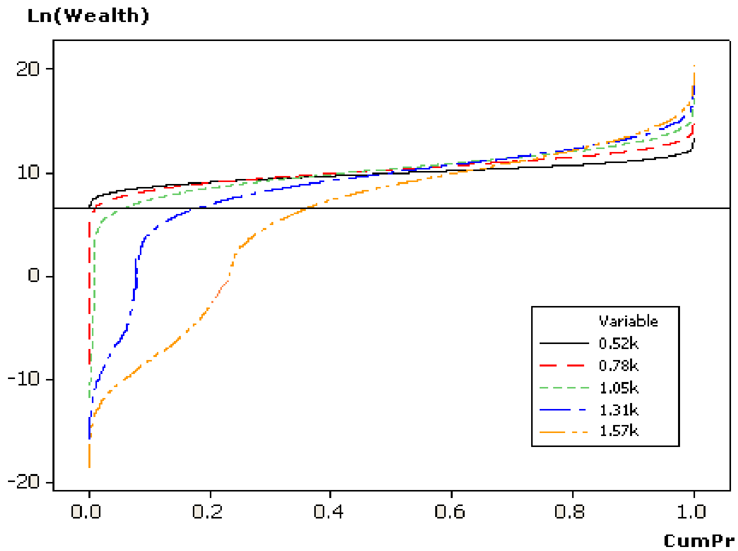


(a) Maximum

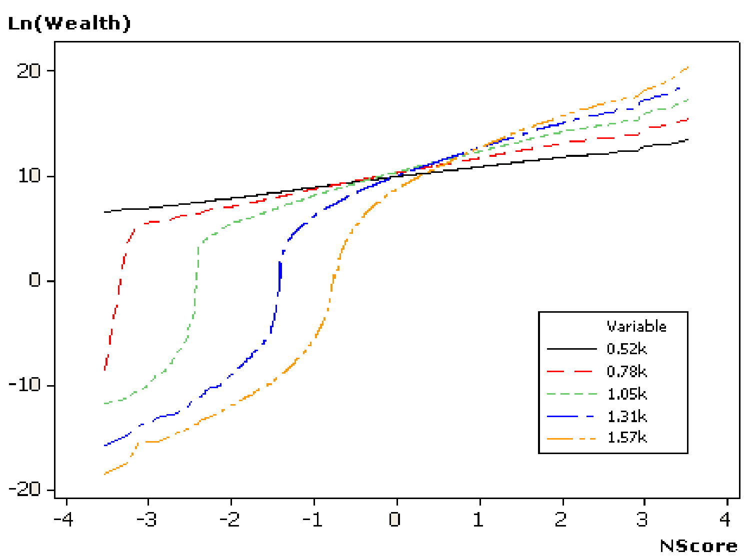


(b) Minimum

Figure 9: Trajectories with Final Wealth Extremes for the Bicksler and Thorp Example II



(a) Inverse Cumulative



(b) Normal Plot

Figure 10: Final Ln(Wealth) Distributions by Fraction for the Bicksler and Thorp Example II

deviation.

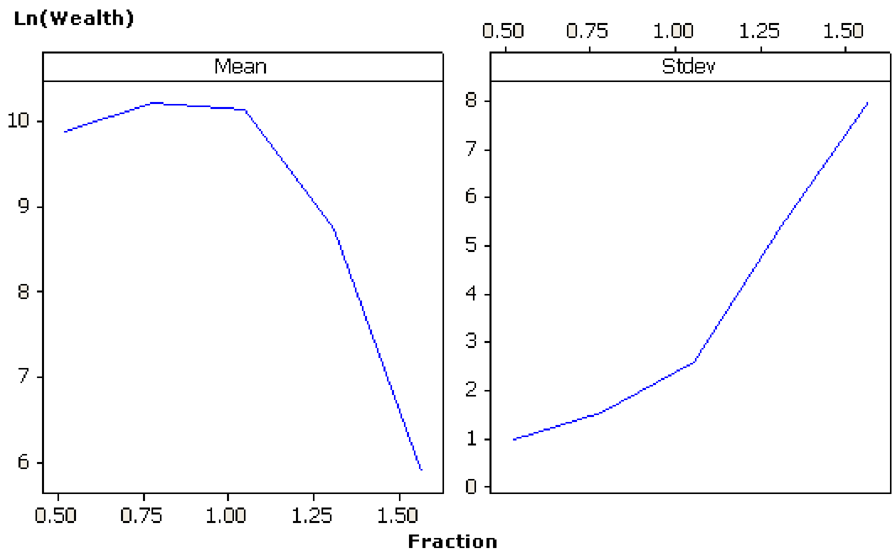


Figure 11: Mean-Std Tradeoff: Bicksler and Thorp Example II

The results in experiment 3 lead to the following conclusions.

1. The statistics describing the end of the horizon ($T = 40$) wealth are again monotone in the fraction of wealth invested in the Kelly portfolio. Specifically (i) the maximum terminal wealth and the mean terminal wealth increase in the Kelly fraction; and (ii) the minimum wealth decreases as the fraction increases and the standard deviation grows as the fraction increases. The growth and decay are pronounced and it is possible to have extremely large losses. The fraction of the Kelly optimal growth strategy exceeds 1 in the most levered strategies and this is very risky. There is a trade-off between return and risk, but the mean for the levered strategies is growing far less than the standard deviation. The disadvantage of leveraged investment is illustrated with the cumulative distributions in Figure 10. The log-normality of final wealth does not hold for the levered strategies.

2. The maximum and minimum final wealth trajectories show the return - risk of levered strategies. The worst and best scenarios are not same for all Kelly fractions. The worst scenario for the most levered strategy shows the rapid decline in wealth. The mean-standard deviation trade-off confirms the extreme riskyness of the aggressive strategies.

Final Comments

The Kelly optimal capital growth investment strategy is an attractive approach to wealth creation. In addition to maximizing the asymptotic rate of long term growth of capital, it avoids bankruptcy and overwhelms any essentially different investment strategy in the long run. See MacLean, Thorp and Ziemba (2010b) for a discussion of the good and bad properties of these strategies. However, automatic use of the Kelly strategy in any investment situation is risky and can be dangerous. It requires some adaptation to the investment environment: rates of return, volatilities, correlation of alternative assets, estimation error, risk aversion preferences, and planning horizon are all important aspects of the investment process. Poundstone's (2005) book, while a very good read, does not explain these important investment aspects and the use of Kelly strategies by advisory firms such as *Morningstar* and *Motley Fools* is flawed. The experiments in this paper represent some of the diversity in the investment environment. By considering the Kelly and its variants we get a concrete look at the plusses and minusses of the capital growth model. The main points from the Bicksler and Thorp (1973) and Ziemba and Hausch (1986) studies are confirmed.

- The wealth accumulated from the full Kelly strategy does not stochastically dominate fractional Kelly wealth. The downside is often much more favorable with a fraction less than one.
- There is a tradeoff of risk and return with the fraction invested in the Kelly portfolio. In cases of large uncertainty, either from intrinsic volatility or estimation error, security is gained by reducing the Kelly investment fraction.
- The full Kelly strategy can be highly levered. While the use of borrowing can be effective in generating large returns on investment, increased leveraging beyond the full Kelly is not warranted as it is growth-security dominated. The returns from over-levered investment are offset by a growing probability of bankruptcy.
- The Kelly strategy is not merely a long term approach. Proper use in the short and medium run can achieve wealth goals while protecting against drawdowns. MacLean, Sanegre, Zhao and Ziemba (2004) and MacLean, Zhao and Ziemba (2009) discuss a strategy to reduce the Kelly fraction to stay above a prespecified wealth path with high probability.

The great economist Paul Samuelson was a long time critic of the Kelly strategy which maximizes the expected logarithm of final wealth, see, for example, Samuelson (1969, 1971, 1979) and Merton and Samuelson (1974). His criticisms are well dealt with in this simulation paper and we see no disagreement with his various analytic points:

1. the Kelly strategy maximizes the asymptotic long run growth of the investor's wealth,

and we agree;

2. the Kelly strategy maximizes expected utility of only logarithmic utility and not necessarily any other utility function, and we agree; and
3. the Kelly strategy always leads to more wealth than any essentially different strategy; this we know from this paper is not true since it is possible to have a large number of very good investments and still lose most of one's fortune.

Samuelson seemed to imply that Kelly proponents thought that the Kelly strategy maximizes for other utility functions but this was neither argued nor implied.

It is true that the expected value of wealth is higher with the Kelly strategy but bad outcomes are very possible.

In correspondence with Ziemba (private correspondence, 2006, 2007, 2008) he seems to feel that half Kelly or $u(w) = -\frac{1}{w}$ explains the data better. We agree that in practice, half Kelly is a toned down version of full Kelly that provides a lot more security to compensate for its loss in long term growth. Samuelson proposes an analysis of three investors $-\frac{1}{w}$, $\log w$ and $w^{\frac{1}{2}}$. In Ziemba (2010) these are explored adding two tail investors αw^{α}, $\alpha \to -\infty$ the safest investor and w, namely $\alpha = 1$, the riskiest investor which span the range of absolute Arrow-Pratt risk aversion from 0 to ∞.

References

Bernoulli, D. (1954). Exposition of a new theory on the measurement of risk (translated by Louise Sommer). *Econometrica 22*, 23–36.

Bicksler, J. L. and E. O. Thorp (1973). The capital growth model: an empirical investigation. *Journal of Financial and Quantitative Analysis 8*(2), 273–287.

Breiman, L. (1961). Optimal gambling system for favorable games. *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability 1*, 63–8.

Browne, S. (1999). The risk and rewards of minimizing shortfall probability. *Journal of Portfolio Management 25*(4), 76–85.

Chopra, V. K. and W. T. Ziemba (1993). The effect of errors in mean, variance and covariance estimates on optimal portfolio choice. *Journal of Portfolio Management 19*, 6–11.

Fuller, J. (2006). Optimize your portfolio with the Kelly formula. morningstar.com, October 6.

Hakansson, N. (1970). Optimal investment and consumption strategies under risk for a class of utility functions. *Econometrica 38*, 587–607.

Hakansson, N. H. and W. T. Ziemba (1995). Capital growth theory. In R. A. Jarrow, V. Maksimovic, and W. T. Ziemba (Eds.), *Finance, Handbooks in OR & MS*, pp. 65–86. North Holland.

Hanoch, G. and H. Levy (1969). The efficiency analysis of choices involving risk. *The Review of Economic Studies 36*, 335–346.

Kelly, Jr., J. R. (1956). A new interpretation of the information rate. *Bell System Technical Journal 35*, 917–926.

Latané, H. (1959). Criteria for choice among risky ventures. *Journal of Political Economy 67*, 144–155.

Lee, E. (2006). How to calculate the Kelly formula. fool.com, October 31.

MacLean, L. C., R. Sanegre, Y. Zhao, and W. T. Ziemba (2004). Capital growth with security. *Journal of Economic Dynamics and Control 28*(4), 937–954.

MacLean, L. C., E. O. Thorp, and W. T. Ziemba (Eds.) (2010a). *The Kelly Capital Growth Investment Criterion: Theory and Practice*. World Scientific Press, Singapore.

MacLean, L. C., E. O. Thorp, and W. T. Ziemba (2010b). Long term capital growth: the good and bad properties of the Kelly criterion criterion. *Quantitative Finance* (September).

MacLean, L. C., Y. Zhao, and W. T. Ziemba (2009). Optimal capital growth with convex loss penalties. Working paper, Dalhousie University.

MacLean, L. C., W. T. Ziemba, and G. Blazenko (1992). Growth versus security in dynamic investment analysis. *Management Science 38*, 1562–85.

MacLean, L. C., W. T. Ziemba, and Y. Li (2005). Time to wealth goals in capital accumulation and the optimal trade-off of growth versus security. *Quantitative Finance 5*(4), 343–357.

McEnally, R. W. (1986). Latané's bequest: The best of portfolio strategies. *Journal of Portfolio Management 12*(2), 21–30.

Merton, R. C. (1990). *Continuous-Time Finance*. Blackwell Publishers, Cambridge, MA.

Merton, R. C. and P. A. Samuelson (1974). Fallacy of the log-normal approximation to optimal portfolio decision-making over many periods. *Journal of Financial Economics 1*, 67–94.

Mulvey, J. M., B. Pauling, and R. E. Madey (2003). Advantages of multiperiod portfolio models. *Journal of Portfolio Management 29*, 35–45.

Pabrai, M. (2007). *The Dhandho Investor*.

Phelps, E. S. (1962). The accumulation of risky capital: A sequential utility analysis. *Econometrica 30*, 729–743.

Poundstone, W. (2005). *Fortune's Formula: The Untold Story of the Scientific System that Beat the Casinos and Wall Street*. Hill and Wang, NY.

Rubinstein, M. (1976). The strong case for the generalized logarithmic utility model as the premier model of financial markets. *Journal of Finance 31*(2), 551–571.

Rubinstein, M. (1991). Continuously rebalanced investment strategies. *Journal of Portfolio Management 18*(1), 78–81.

Samuelson, P. A. (1969). Lifetime portfolio selection by dynamic stochastic programming. *Review of Economics and Statistics 51*, 239–246.

Samuelson, P. A. (1971). The fallacy of maximizing the geometric mean in long sequences of investing or gambling. *Proceedings National Academy of Science 68*, 2493–2496.

Samuelson, P. A. (1979). Why we should not make mean log of wealth big though years to act are long. *Journal of Banking and Finance 3*, 305–307.

Samuelson, P. A. (various). Letters to William T. Ziemba. Private Correspondence, December 13, 2006, May 7, 2007, May 12, 2008.

Siegel, J. J. (2002). *Stocks for the long run*. Wiley.

Stutzer, M. (2000). A portfolio performance index. *Financial Analysts Journal 56*(3), 52–61.

Stutzer, M. (2004). Asset allocation without unobservable parameters. *Financial Analysts Journal 60*(5), 38–51.

Thorp, E. O. (2006). The Kelly criterion in blackjack, sports betting and the stock market. In S. A. Zenios and W. T. Ziemba (Eds.), *Handbook of Asset and Liability Management, Volume 1*, pp. 387–428. North Holland.

Thorp, E. O. (2010). Understanding the Kelly criterion. In L. C. MacLean, E. O. Thorp, and W. T. Ziemba (Eds.), *The Kelly Capital Growth Investment Criterion: Theory and Practice*. World Scientific Press, Singapore.

Wilcox, J. (2003a). Harry Markowitz and the discretionary wealth hypothesis. *Journal of Portfolio Management 29*(Spring), 58–65.

Wilcox, J. (2003b). Risk management: survival of the fittest. Wilcox Investment Inc.

Wilcox, J. (2005). A better paradigm for finance. *Finance Letters 3*(1), 5–11.

Ziemba, R. E. S. and W. T. Ziemba (2007). *Scenarios for Risk Management and Global Investment Strategies*. Wiley.

Ziemba, W. T. (2005). The symmetric downside risk Sharpe ratio and the evaluation of great investors and speculators. *Journal of Portfolio Management Fall*, 108–122.

Ziemba, W. T. (2010). A tale of five investors: response to Paul A. Samuelson letters. Working Paper, University of Oxford.

Ziemba, W. T. and D. B. Hausch (1986). *Betting at the Racetrack*. Dr Z Investments, San Luis Obispo, CA.

Ziemba, W. T. and D. B. Hausch (1987). *Dr Z's Beat the Racetrack*. William Morrow.

Ziemba, W. T. and R. G. Vickson (Eds.) (1975). *Stochastic optimization models in finance*. Academic Press, NY.

Ziemba, W. T. and R. G. Vickson (Eds.) (2006). *Stochastic optimization models in finance* (2 ed.). World Scientific.